

Exponential laws for spaces of differentiable functions on topological groups

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Abstract

Smooth functions $f : G \rightarrow E$ from a topological group G to a locally convex space E were considered by Riss (1953), Boseck, Czichowski and Rudolph (1981), Beltiță and Nicolae (2015), and others, in varying degrees of generality. The space $C^\infty(G, E)$ of such functions carries a natural topology, the compact-open C^∞ -topology. For topological groups G and H , we show that $C^\infty(G \times H, E) \cong C^\infty(G, C^\infty(H, E))$ as a locally convex space, whenever both G and H are metrizable or both G and H are locally compact. Likewise, $C^k(G, C^l(H, E))$ can be identified with a suitable space of functions on $G \times H$.

1 Introduction

Exponential laws of the form $C^\infty(M \times N, E) \cong C^\infty(M, C^\infty(N, E))$ for spaces of vector-valued smooth functions on manifolds are essential tools in infinite-dimensional calculus and infinite-dimensional Lie theory (cf. works by Kriegl and Michor [10], Kriegl, Michor and Rainer [11], Alzaareer and Schmeding [1], Glöckner [5], Glöckner and Neeb [6], Neeb and Wagemann [12], and others). Stimulated by recent research by Beltiță and Nicolae [2], we provide exponential laws for function spaces on topological groups.

Let G be a topological group, $U \subseteq G$ be an open subset, $f : U \rightarrow E$ be a function to a locally convex space and $\mathfrak{L}(G) := \text{Hom}_{cts}(\mathbb{R}, G)$ be the set of continuous one-parameter subgroups $\gamma : \mathbb{R} \rightarrow G$, endowed with the compact-open topology. For $x \in U$ and $\gamma \in \mathfrak{L}(G)$ let us write

$$D_\gamma f(x) := \lim_{t \rightarrow 0} \frac{1}{t} (f(x \cdot \gamma(t)) - f(x))$$

if the limit exists. Following Riss [14] and Boseck et al. [3], we say that f is C^k (where $k \in \mathbb{N}_0 \cup \{\infty\}$) if f is continuous, the iterated derivatives

$$d^{(i)} f(x, \gamma_1, \dots, \gamma_i) := (D_{\gamma_i} \cdots D_{\gamma_1} f)(x)$$

exist for all $x \in U$, $i \in \mathbb{N}$ with $i \leq k$ and $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$, and the maps $d^{(i)} f : U \times \mathfrak{L}(G)^i \rightarrow E$ so obtained are continuous. We endow the space $C^k(U, E)$ of all C^k -maps $f : U \rightarrow E$ with the compact-open C^k -topology (recalled in Definition 2.3). If G and H are topological groups and $f : G \times H \rightarrow E$ is C^∞ , then

$f^\vee(x) := f(x, \bullet) \in C^\infty(H, E)$ for all $x \in G$. With a view towards universal enveloping algebras, Beltiță and Nicolae [2] verified that $f^\vee \in C^\infty(G, C^\infty(H, E))$ and showed that the linear map

$$\Phi : C^\infty(G \times H, E) \rightarrow C^\infty(G, C^\infty(H, E)), \quad f \mapsto f^\vee$$

is a topological embedding.

Recall that a Hausdorff space X is called a $k_{\mathbb{R}}$ -space if functions $f : X \rightarrow \mathbb{R}$ are continuous if and only if $f|_K$ is continuous for each compact subset $K \subseteq X$. We obtain the following criterion for surjectivity of Φ :

Theorem (A). *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H , and E be a locally convex space. If $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_{\mathbb{R}}$ -space for all $i, j \in \mathbb{N}_0$, then*

$$\Phi : C^\infty(U \times V, E) \rightarrow C^\infty(U, C^\infty(V, E)), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces.

The condition is satisfied, for example, if both G and H are locally compact or both G and H are metrizable (see Corollary 3.5).

Generalizing the case of open subsets U and V in locally convex spaces treated by Alzaareer and Schmeding [1] and Glöckner and Neeb [6], we introduce $C^{k,l}$ -functions $f : U \times V \rightarrow E$ on open subsets $U \subseteq G$ and $V \subseteq H$ of topological groups with separate degrees $k, l \in \mathbb{N}_0 \cup \{\infty\}$ of differentiability in the two variables, and a natural topology on the space $C^{k,l}(U \times V, E)$ of such maps (see Definition 2.4 for details). Theorem (A) is a consequence of the following result:

Theorem (B). *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H , let E be a locally convex space and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. If $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_{\mathbb{R}}$ -space for all $i, j \in \mathbb{N}_0$ with $i \leq k, j \leq l$, then*

$$\Phi : C^{k,l}(U \times V, E) \rightarrow C^k(U, C^l(V, E)), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces.

Notation: All topological spaces are assumed Hausdorff. We call a map $f : X \rightarrow Y$ between topological spaces X and Y a *topological embedding* if f is a homeomorphism onto its image (it is known that an injective map f is a topological embedding if and only if the topology on X is initial with respect to f , that is, X carries the coarsest topology making f continuous).

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2 Differentiability of mappings on topological groups

Definition 2.1. Let G be a topological group, a *one-parameter subgroup* is a group homomorphism $\gamma : \mathbb{R} \rightarrow G$. We denote by $\mathfrak{L}(G) := \text{Hom}_{cts}(\mathbb{R}, G)$ the

set of all continuous one-parameter subgroups, endowed with the compact-open topology.

Remark 2.2. If $\gamma \in \mathfrak{L}(G)$ and $\phi : G \rightarrow H$ is a continuous group homomorphism, then $\phi \circ \gamma \in \mathfrak{L}(H)$ and the map $\mathfrak{L}(\phi) : \mathfrak{L}(G) \rightarrow \mathfrak{L}(H), \gamma \mapsto \phi \circ \gamma$ is continuous (cf. [6, Appendix A.5], see also [4, Appendix B]).

Further, for $\psi = (\gamma, \eta) \in C(\mathbb{R}, G \times H)$ it is easy to see that $\psi \in \mathfrak{L}(G \times H)$ if and only if $\gamma \in \mathfrak{L}(G)$ and $\eta \in \mathfrak{L}(H)$. Moreover, the natural map $(\mathfrak{L}(\text{pr}_1), \mathfrak{L}(\text{pr}_2)) : \mathfrak{L}(G \times H) \rightarrow \mathfrak{L}(G) \times \mathfrak{L}(H)$ (where $\text{pr}_1 : G \times H \rightarrow G$, $\text{pr}_2 : G \times H \rightarrow H$ are the coordinate projections) is a homeomorphism (cf. [6, Appendix A.5], [4, Appendix B]).

Now, we define the notion of differentiability along one-parameter subgroups of vector-valued functions on topological groups:

Definition 2.3. Let $U \subseteq G$ be an open subset of a topological group G and E be a locally convex space. For a map $f : U \rightarrow E$, $x \in U$ and $\gamma \in \mathfrak{L}(G)$ we define

$$d^{(1)}f(x, \gamma) := df(x, \gamma) := D_\gamma f(x) := \lim_{t \rightarrow 0} \frac{1}{t} (f(x \cdot \gamma(t)) - f(x))$$

if the limit exists.

We call f a C^k -map for $k \in \mathbb{N}$ if f is continuous and for each $x \in U$, $i \in \mathbb{N}$ with $i \leq k$ and $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$ the iterative derivatives

$$d^{(i)}f(x, \gamma_1, \dots, \gamma_i) := (D_{\gamma_i} \cdots D_{\gamma_1} f)(x)$$

exist and define continuous maps

$$d^{(i)}f : U \times \mathfrak{L}(G)^i \rightarrow E, \quad (x, \gamma_1, \dots, \gamma_i) \mapsto (D_{\gamma_i} \cdots D_{\gamma_1} f)(x).$$

If f is C^k for each $k \in \mathbb{N}$, then we call f a C^∞ -map or *smooth*. Further, we call continuous maps C^0 and denote $d^{(0)}f := f$.

The set of all C^k -maps $f : U \rightarrow E$ will be denoted by $C^k(U, E)$ and we endow it with the initial topology with respect to the family $(d^{(i)})_{i \in \mathbb{N}_0, i \leq k}$ of maps

$$d^{(i)} : C^k(U, E) \rightarrow C(U \times \mathfrak{L}(G)^i, E)_{c.o.}, \quad f \mapsto d^{(i)}f$$

(where the right-hand side is equipped with the compact-open topology) turning $C^k(U, E)$ into a Hausdorff locally convex space. (This topology is known as the *compact-open C^k -topology*.)

Definition 2.4. Let $U \subseteq G$ and $V \subseteq H$ be open subsets of topological groups G and H , let E be a locally convex space. For a map $f : U \times V \rightarrow E$, $x \in U$, $y \in V$, $\gamma \in \mathfrak{L}(G)$ and $\eta \in \mathfrak{L}(H)$ we define

$$d^{(1,0)}f(x, y, \gamma) := D_{(\gamma, 0)}f(x, y) := \lim_{t \rightarrow 0} \frac{1}{t} (f(x \cdot \gamma(t), y) - f(x, y))$$

and

$$d^{(0,1)}f(x, y, \eta) := D_{(0,\eta)}f(x, y) := \lim_{t \rightarrow 0} \frac{1}{t}(f(x, y \cdot \eta(t)) - f(x, y))$$

whenever the limits exist.

We call a continuous map $f : U \times V \rightarrow E$ a $C^{k,l}$ -map for $k, l \in \mathbb{N}_0 \cup \{\infty\}$ if the derivatives

$$d^{(i,j)}f(x, y, \gamma_1, \dots, \gamma_i, \eta_1, \dots, \eta_j) := (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} D_{(0,\eta_j)} \cdots D_{(0,\eta_1)}f)(x, y)$$

exist for all $x \in U$, $y \in V$, $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ and $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$, $\eta_1, \dots, \eta_j \in \mathfrak{L}(H)$, and define continuous functions

$$\begin{aligned} d^{(i,j)}f : U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j &\rightarrow E \\ (x, y, \gamma_1, \dots, \gamma_i, \eta_1, \dots, \eta_j) &\mapsto (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} D_{(0,\eta_j)} \cdots D_{(0,\eta_1)}f)(x, y). \end{aligned}$$

We endow the space $C^{k,l}(U \times V, E)$ of all $C^{k,l}$ -functions $f : U \times V \rightarrow E$ with the Hausdorff locally convex initial topology with respect to the family $(d^{(i,j)})_{i,j \in \mathbb{N}_0, i \leq k, j \leq l}$ of maps

$$d^{(i,j)} : C^{k,l}(U \times V, E) \rightarrow C(U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j, E)_{c.o.}, \quad f \mapsto d^{(i,j)}f,$$

where the right-hand side is equipped with the compact-open topology. (The so obtained topology on $C^{k,l}(U \times V, E)$ is called the *compact-open $C^{k,l}$ -topology*.)

Remark 2.5. If $k = 0$ or $l = 0$, then the definition of $C^{k,l}$ -maps $f : U \times V \rightarrow E$ also makes sense if U or V , respectively, is any Hausdorff topological space. All further results for $C^{k,l}$ -maps on topological groups carry over to this situation.

Remark 2.6. Simple computations show that for $k \geq 1$ a map $f : U \rightarrow E$ is C^k if and only if f is C^1 and $df : U \times \mathfrak{L}(G) \rightarrow E$ is $C^{k-1,0}$, in this case we have $d^{(i,0)}(df) = d^{(i+1)}f$ for all $i \in \mathbb{N}$ with $i \leq k-1$.

Similarly, we can show that a map $f : U \times V \rightarrow E$ is $C^{k,0}$ if and only if f is $C^{1,0}$ and $d^{(1,0)}f : U \times (V \times \mathfrak{L}(G)) \rightarrow E$ is $C^{k-1,0}$ with differentials $d^{(i,0)}(d^{(1,0)}f) = d^{(i+1,0)}f$ for all i as above.

Further, if a map $f : U \times V \rightarrow E$ is $C^{k,l}$, then for each $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ and fixed $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$, $\eta_1, \dots, \eta_j \in \mathfrak{L}(H)$ the map

$$D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} D_{(0,\eta_j)} \cdots D_{(0,\eta_1)}f : U \times V \rightarrow E$$

is $C^{k,l-j}$ if $i = 0$, and $C^{k-i,0}$ otherwise.

We warn the reader that the full statement of Schwarz' Theorem does not carry over to non-abelian topological groups; for a function $f : G \rightarrow \mathbb{R}$ and $\gamma, \eta \in \mathfrak{L}(G)$ such that $D_\gamma f, D_\eta f, D_\gamma D_\eta f : U \rightarrow \mathbb{R}$ are continuous functions it may happen that $D_\gamma D_\eta f \neq D_\eta D_\gamma f$ (see Example A.6). Nevertheless, we can prove the following restricted version of Schwarz' Theorem for $C^{k,l}$ -maps:

Proposition 2.7. *Let $U \subseteq G$ and $V \subseteq H$ be open subsets of topological groups G and H , let E be a locally convex space and $f : U \times V \rightarrow E$ be a $C^{k,l}$ -map for some $k, l \in \mathbb{N} \cup \{\infty\}$. Then the derivatives*

$$(D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} f)(x, y)$$

exist for all $(x, y) \in U \times V$, $i, j \in \mathbb{N}$ with $i \leq k$, $j \leq l$ and $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$, $\eta_1, \dots, \eta_j \in \mathfrak{L}(H)$ and we have

$$\begin{aligned} (D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} f)(x, y) \\ = (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} f)(x, y). \end{aligned}$$

Proof. First we prove the assertion for $j = 1$ by induction on i .

Induction start: Let $(x, y) \in U \times V$, $\gamma \in \mathfrak{L}(G)$ and $\eta \in \mathfrak{L}(H)$. For suitable $\varepsilon, \delta > 0$ we define the continuous map

$$h :] - \varepsilon, \varepsilon[\times] - \delta, \delta[\rightarrow E, \quad (s, t) \mapsto f(x \cdot \gamma(s), y \cdot \eta(t)),$$

and obtain the partial derivatives of h via

$$\begin{aligned} \frac{\partial h}{\partial s}(s, t) &= \lim_{r \rightarrow 0} \frac{1}{r} (h(s + r, t) - h(s, t)) \\ &= \lim_{r \rightarrow 0} \frac{1}{r} (f(x \cdot \gamma(s) \cdot \gamma(r), y \cdot \eta(t)) - f(x \cdot \gamma(s), y \cdot \eta(t))) \\ &= D_{(\gamma,0)} f(x \cdot \gamma(s), y \cdot \eta(t)), \end{aligned}$$

and analogously,

$$\frac{\partial h}{\partial t}(s, t) = D_{(0,\eta)} f(x \cdot \gamma(s), y \cdot \eta(t))$$

and

$$\frac{\partial^2 h}{\partial s \partial t}(s, t) = (D_{(\gamma,0)} D_{(0,\eta)} f)(x \cdot \gamma(s), y \cdot \eta(t)).$$

The obtained maps $\frac{\partial h}{\partial s}$, $\frac{\partial h}{\partial t}$ and $\frac{\partial^2 h}{\partial s \partial t}$ are continuous, hence we apply [6, Lemma 1.3.18], which states that in this case also the partial derivative $\frac{\partial^2 h}{\partial t \partial s}$ exists and coincides with $\frac{\partial^2 h}{\partial s \partial t}$. Therefore, we have

$$\begin{aligned} (D_{(\gamma,0)} D_{(0,\eta)} f)(x, y) &= \frac{\partial^2 h}{\partial s \partial t}(0, 0) = \frac{\partial^2 h}{\partial t \partial s}(0, 0) = \lim_{r \rightarrow 0} \frac{1}{r} \left(\frac{\partial h}{\partial s}(0, r) - \frac{\partial h}{\partial s}(0, 0) \right) \\ &= \lim_{r \rightarrow 0} \frac{1}{r} (D_{(\gamma,0)} f(x, y \cdot \eta(r)) - D_{(\gamma,0)} f(x, y)) \\ &= (D_{(0,\eta)} D_{(\gamma,0)} f)(x, y). \end{aligned}$$

Thus the assertion holds for $i = 1$.

Induction step: Now, let $2 \leq i \leq k$, $(x, y) \in U \times V$, $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$ and $\eta \in \mathfrak{L}(H)$. Consider the map

$$g_1 : U \times V \rightarrow E, \quad (x, y) \mapsto (D_{(\gamma_{i-1}, 0)} \cdots D_{(\gamma_1, 0)} f)(x, y),$$

which is $C^{1,0}$ (see Remark 2.6). Further, g_1 is $C^{0,1}$, because

$$\begin{aligned} D_{(0, \eta)} g_1(x, y) &= (D_{(0, \eta)} D_{(\gamma_{i-1}, 0)} \cdots D_{(\gamma_1, 0)} f)(x, y) \\ &= (D_{(\gamma_{i-1}, 0)} \cdots D_{(\gamma_1, 0)} D_{(0, \eta)} f)(x, y), \end{aligned}$$

by the induction hypothesis, and we see that

$$(D_{(\gamma_i, 0)} D_{(0, \eta)} g_1)(x, y) = (D_{(\gamma_i, 0)} D_{(\gamma_{i-1}, 0)} \cdots D_{(\gamma_1, 0)} D_{(0, \eta)} f)(x, y),$$

whence g_1 is $C^{1,1}$. By the induction start, the derivative $(D_{(0, \eta)} D_{(\gamma_i, 0)} g_1)(x, y)$ exists and equals $(D_{(\gamma_i, 0)} D_{(0, \eta)} g_1)(x, y)$, thus we get

$$\begin{aligned} (D_{(\gamma_i, 0)} \cdots D_{(\gamma_1, 0)} D_{(0, \eta)} f)(x, y) &= (D_{(\gamma_i, 0)} D_{(0, \eta)} g_1)(x, y) \\ &= (D_{(0, \eta)} D_{(\gamma_i, 0)} g_1)(x, y) \\ &= (D_{(0, \eta)} D_{(\gamma_i, 0)} \cdots D_{(\gamma_1, 0)} f)(x, y). \end{aligned}$$

Hence the assertion holds for $j = 1$.

Now, let $2 \leq j \leq l$, $1 \leq i \leq k$, $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$, $\eta_1, \dots, \eta_j \in \mathfrak{L}(H)$ and $(x, y) \in U \times V$. By Remark 2.6, the map

$$g_2 : U \times V \rightarrow E, \quad (x, y) \mapsto (D_{(0, \eta_{j-1})} \cdots D_{(0, \eta_1)} f)(x, y)$$

is $C^{k,1}$, whence we have

$$\begin{aligned} &(D_{(0, \eta_j)} D_{(\gamma_i, 0)} \cdots D_{(\gamma_1, 0)} D_{(0, \eta_{j-1})} \cdots D_{(0, \eta_1)} f)(x, y) \\ &= (D_{(0, \eta_j)} D_{(\gamma_i, 0)} \cdots D_{(\gamma_1, 0)} g_2)(x, y) \\ &= (D_{(\gamma_i, 0)} \cdots D_{(\gamma_1, 0)} D_{(0, \eta_j)} g_2)(x, y) \\ &= (D_{(\gamma_i, 0)} \cdots D_{(\gamma_1, 0)} D_{(0, \eta_j)} \cdots D_{(0, \eta_1)} f)(x, y), \end{aligned} \tag{1}$$

using the first part of the proof. But we also have

$$\begin{aligned} &(D_{(0, \eta_j)} D_{(\gamma_i, 0)} \cdots D_{(\gamma_1, 0)} D_{(0, \eta_{j-1})} \cdots D_{(0, \eta_1)} f)(x, y) \\ &= (D_{(0, \eta_j)} D_{(0, \eta_{j-1})} \cdots D_{(0, \eta_1)} D_{(\gamma_i, 0)} \cdots D_{(\gamma_1, 0)} f)(x, y), \end{aligned} \tag{2}$$

by induction, whence (2) equals (1), that is

$$\begin{aligned} &(D_{(0, \eta_j)} \cdots D_{(0, \eta_1)} D_{(\gamma_i, 0)} \cdots D_{(\gamma_1, 0)} f)(x, y) \\ &= (D_{(\gamma_i, 0)} \cdots D_{(\gamma_1, 0)} D_{(0, \eta_j)} \cdots D_{(0, \eta_1)} f)(x, y), \end{aligned}$$

and the proof is finished. \square

Corollary 2.8. *Let $U \subseteq G$ and $V \subseteq H$ be open subsets of topological groups G and H , let E be a locally convex space and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. A map $f : U \times V \rightarrow E$ is $C^{k,l}$ if and only if the map*

$$g : V \times U \rightarrow E, \quad (y, x) \mapsto f(x, y)$$

is $C^{l,k}$. Moreover, we have

$$d^{(j,i)}g(y, x, \eta_1, \dots, \eta_j, \gamma_1, \dots, \gamma_i) = d^{(i,j)}f(x, y, \gamma_1, \dots, \gamma_i, \eta_1, \dots, \eta_j)$$

for each $x \in U$, $y \in V$, $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ and $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$, $\eta_1, \dots, \eta_j \in \mathfrak{L}(H)$.

Proof. First, we assume that $l = 0$, that is, $f : U \times V \rightarrow E$ is $C^{k,0}$. Then for $x \in U$, $y \in V$ and $\gamma \in \mathfrak{L}(G)$ we have

$$\begin{aligned} d^{(1,0)}f(x, y, \gamma) &= \lim_{t \rightarrow 0} \frac{1}{t} (f(x \cdot \gamma(t), y) - f(x, y)) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (g(y, x \cdot \gamma(t)) - g(y, x)) = d^{(0,1)}g(y, x, \gamma), \end{aligned}$$

and similarly we get $d^{(0,i)}g(y, x, \gamma_1, \dots, \gamma_i) = d^{(i,0)}f(x, y, \gamma_1, \dots, \gamma_i)$ for each $i \in \mathbb{N}$ with $i \leq k$ and $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$. The obtained maps $d^{(0,i)}g : V \times U \times \mathfrak{L}(G)^i \rightarrow E$ are obviously continuous, hence g is $C^{0,k}$. The other implication, as well as the case $k = 0$, can be proven analogously.

If $k, l \geq 1$, then the assertion follows immediately from Proposition 2.7. \square

Remark 2.9. Using Remark 2.6 and Corollary 2.8, we can easily show that if $f : U \times V \rightarrow E$ is $C^{k,l}$, then for all $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ and fixed $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$, $\eta_1, \dots, \eta_j \in \mathfrak{L}(H)$ the maps

$$D_{(\gamma_i, 0)} \cdots D_{(\gamma_1, 0)} D_{(0, \eta_j)} \cdots D_{(0, \eta_1)} f : U \times V \rightarrow E$$

are $C^{k-i, l-j}$.

The following lemma will be useful:

Lemma 2.10. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H , let E, F be locally convex spaces, $\lambda : E \rightarrow F$ be a continuous and linear map and $k, l \in \mathbb{N}_0 \cup \{\infty\}$.*

(a) If $f : U \rightarrow E$ is a C^k -map, then the map $\lambda \circ f : U \rightarrow F$ is C^k .

(b) If $f : U \times V \rightarrow E$ is a $C^{k,l}$ -map, then the map $\lambda \circ f : U \times V \rightarrow F$ is $C^{k,l}$.

Proof. To prove (a), let $x \in U$, $\gamma \in \mathfrak{L}(G)$ and $t \neq 0$ small enough, then we have

$$\frac{\lambda(f(x \cdot \gamma(t))) - \lambda(f(x))}{t} = \lambda \left(\frac{f(x \cdot \gamma(t)) - f(x)}{t} \right) \rightarrow \lambda(df(x, \gamma)),$$

as $t \rightarrow 0$, because λ is assumed linear and continuous. Therefore, the derivative $d(\lambda \circ f)(x, \gamma)$ exists and we have $d(\lambda \circ f)(x, \gamma) = (\lambda \circ df)(x, \gamma)$.

Proceeding similarly, for each $i \in \mathbb{N}$ with $i \leq k$, $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$ we obtain $d^{(i)}(\lambda \circ f)(x, \gamma_1, \dots, \gamma_i) = (\lambda \circ d^{(i)}f)(x, \gamma_1, \dots, \gamma_i)$. Since each of the obtained maps $d^{(i)}(\lambda \circ f) = \lambda \circ d^{(i)}f : U \times \mathfrak{L}(G)^i \rightarrow F$ is continuous, we see that the map $\lambda \circ f$ is C^k .

Analogously, assertion (b) can be proved showing that for each $i, j \in \mathbb{N}_0$ with $i \leq k, j \leq l$ we have $d^{(i,j)}(\lambda \circ f) = \lambda \circ d^{(i,j)}f$. \square

Let us introduce the following notation (the analogue for C^1 -maps is Lemma A.3):

Lemma 2.11. *Let $U \subseteq G, V \subseteq H$ be open subsets of topological groups G and H , let E be a locally convex space. A continuous map $f : U \times V \rightarrow E$ is $C^{1,0}$ if and only if there exists a continuous map*

$$f^{[1,0]} : U^{[1]} \times V \rightarrow E,$$

where

$$U^{[1]} := \{(x, \gamma, t) \in U \times \mathfrak{L}(G) \times \mathbb{R} : x \cdot \gamma(t) \in U\},$$

such that

$$f^{[1,0]}(x, \gamma, t, y) = \frac{1}{t}(f(x \cdot \gamma(t), y) - f(x, y))$$

for each $(x, \gamma, t, y) \in U^{[1]} \times V$ with $t \neq 0$.

In this case we have $d^{(1,0)}f(x, y, \gamma) = f^{[1,0]}(x, \gamma, 0, y)$ for all $x \in U, y \in V$ and $\gamma \in \mathfrak{L}(G)$.

Proof. First, assume that the map $f^{[1,0]}$ exists and is continuous. Then for $x \in U, y \in V, \gamma \in \mathfrak{L}(G)$ and $t \neq 0$ small enough we have

$$\frac{1}{t}(f(x \cdot \gamma(t), y) - f(x, y)) = f^{[1,0]}(x, \gamma, t, y) \rightarrow f^{[1,0]}(x, \gamma, 0, y)$$

as $t \rightarrow 0$. Hence $d^{(1,0)}f(x, y, \gamma)$ exists and is given by $f^{[1,0]}(x, \gamma, 0, y)$, whence the map

$$d^{(1,0)}f : U \times V \times \mathfrak{L}(G) \rightarrow E, \quad (x, y, \gamma) \mapsto f^{[1,0]}(x, \gamma, 0, y)$$

is continuous. Thus f is $C^{1,0}$.

Conversely, let f be a $C^{1,0}$ -map. Then we define

$$f^{[1,0]} : U^{[1]} \times V \rightarrow E, \quad f^{[1,0]}(x, \gamma, t, y) := \begin{cases} \frac{f(x \cdot \gamma(t), y) - f(x, y)}{t} & \text{if } t \neq 0 \\ d^{(1,0)}f(x, y, \gamma) & \text{if } t = 0. \end{cases}$$

Since f is continuous, the map $f^{[1,0]}$ is continuous at each (x, γ, t, y) with $t \neq 0$. Given $x_0 \in U$ and $\gamma_0 \in \mathfrak{L}(G)$, we have $(x_0, \gamma_0, 0) \in U^{[1]}$; let $W := U_{x_0} \times U_{\gamma_0} \times]-\varepsilon, \varepsilon[\subseteq U^{[1]}$ be an open neighborhood of $(x_0, \gamma_0, 0)$ in $U^{[1]}$, where $U_{x_0} \subseteq U$ and $U_{\gamma_0} \subseteq \mathfrak{L}(G)$ are open neighborhoods of x_0 and γ_0 , respectively, and $\varepsilon > 0$. Now, for fixed $(x, \gamma, y) \in U_{x_0} \times U_{\gamma_0} \times V$ we define the continuous curve

$$h :]-\varepsilon, \varepsilon[\rightarrow E, \quad h(t) := f(x \cdot \gamma(t), y).$$

Then for $t \in]-\varepsilon, \varepsilon[$, $s \neq 0$ with $t + s \in]-\varepsilon, \varepsilon[$ we have

$$\begin{aligned} \frac{h(t+s) - h(t)}{s} &= \frac{f(x \cdot \gamma(t+s), y) - f(x \cdot \gamma(t), y)}{s} \\ &= \frac{f(x \cdot \gamma(t) \cdot \gamma(s), y) - f(x \cdot \gamma(t), y)}{s} \rightarrow d^{(1,0)} f(x \cdot \gamma(t), y, \gamma) \end{aligned}$$

as $s \rightarrow 0$. Thus, the derivative $h'(t)$ exists and is given by $d^{(1,0)} f(x \cdot \gamma(t), y, \gamma)$. The so obtained map $h' :]-\varepsilon, \varepsilon[\rightarrow E$ is continuous, hence h is a C^1 -curve (see [6] for details on C^1 -curves with values in locally convex spaces and also on weak integrals which we use in the next step). We use the Fundamental Theorem of Calculus ([6, Proposition 1.1.5]) and obtain for $t \neq 0$

$$\begin{aligned} f^{[1,0]}(x, \gamma, t, y) &= \frac{1}{t} (f(x \cdot \gamma(t), y) - f(x, y)) = \frac{1}{t} (h(t) - h(0)) \\ &= \frac{1}{t} \int_0^t h'(\tau) d\tau = \frac{1}{t} \int_0^t d^{(1,0)} f(x \cdot \gamma(\tau), y, \gamma) d\tau \\ &= \frac{1}{t} \int_0^1 t d^{(1,0)} f(x \cdot \gamma(tu), y, \gamma) du = \int_0^1 d^{(1,0)} f(x \cdot \gamma(tu), y, \gamma) du. \end{aligned}$$

But if $t = 0$, then

$$\int_0^1 d^{(1,0)} f(x \cdot \gamma(0), y, \gamma) du = d^{(1,0)} f(x, y, \gamma) = f^{[1,0]}(x, \gamma, 0, y),$$

hence

$$f^{[1,0]}(x, \gamma, t, y) = \int_0^1 d^{(1,0)} f(x \cdot \gamma(tu), y, \gamma) du$$

for all $(x, \gamma, t, y) \in W \times V$. Since the map

$$W \times V \times [0, 1] \rightarrow E, \quad (x, \gamma, t, y, u) \mapsto d^{(1,0)} f(x \cdot \gamma(tu), y, \gamma)$$

is continuous, also the parameter-dependent integral

$$W \times V \rightarrow E, \quad (x, \gamma, t, y) \mapsto f^{[1,0]}(x, \gamma, t, y) = \int_0^1 d^{(1,0)} f(x \cdot \gamma(tu), y, \gamma) du$$

is continuous (by [6, Lemma 1.1.11]), in particular in $(x_0, \gamma_0, 0, y)$. Consequently, $f^{[1,0]}$ is continuous. \square

The following two propositions provide a relation between C^k - and $C^{k,l}$ -maps on products of topological groups (a version can also be found in [3]), in particular, we will conclude that $C^{\infty,\infty}(U \times V, E) \cong C^\infty(U \times V, E)$ as topological vector spaces (Corollary 2.14).

Proposition 2.12. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H , let E be a locally convex space and $k \in \mathbb{N}_0 \cup \{\infty\}$. If $f : U \times V \rightarrow E$ is $C^{k,k}$, then f is C^k .*

Moreover, the inclusion map

$$\Psi : C^{k,k}(U \times V, E) \rightarrow C^k(U \times V, E), \quad f \mapsto f$$

is continuous and linear.

Proof. The case $k = 0$ is trivial. For $k \geq 1$, we show by induction on $i \in \mathbb{N}$ with $i \leq k$ that for all $(x, y) \in U \times V$, $(\gamma_1, \eta_1), \dots, (\gamma_i, \eta_i) \in \mathfrak{L}(G \times H)$ the derivatives of f are given by

$$\begin{aligned} d^{(i)} f((x, y), (\gamma_1, \eta_1), \dots, (\gamma_i, \eta_i)) \\ = \sum_{j=0}^i \sum_{I_{j,i}} d^{(j, i-j)} f(x, y, \gamma_{r_1}, \dots, \gamma_{r_j}, \eta_{s_1}, \dots, \eta_{s_{i-j}}) \end{aligned} \quad (3)$$

where $I_{j,i} := \{r_1, \dots, r_j\} \dot{\cup} \{s_1, \dots, s_{i-j}\} = \{1, \dots, i\}$.

Induction start: Let $(x, y) \in U \times V$ and $(\gamma, \eta) \in \mathfrak{L}(G \times H)$, that is, $\gamma \in \mathfrak{L}(G)$ and $\eta \in \mathfrak{L}(H)$, see Remark 2.2. For $t \neq 0$ small enough we have

$$\begin{aligned} & \frac{f((x, y) \cdot (\gamma(t), \eta(t))) - f(x, y)}{t} \\ &= \frac{f(x \cdot \gamma(t), y \cdot \eta(t)) - f(x, y)}{t} \\ &= \frac{f(x \cdot \gamma(t), y \cdot \eta(t)) - f(x \cdot \gamma(t), y)}{t} + \frac{f(x \cdot \gamma(t), y) - f(x, y)}{t} \\ &= \frac{g(y \cdot \eta(t), x \cdot \gamma(t)) - g(y, x \cdot \gamma(t))}{t} + \frac{f(x \cdot \gamma(t), y) - f(x, y)}{t}, \end{aligned}$$

where g is the map $g : V \times U \rightarrow E$, $(y, x) \mapsto f(x, y)$. By Corollary 2.8, the map g is $C^{1,1}$, whence the map $g^{[1,0]}$ is defined and continuous, as well as $f^{[1,0]}$ (see Lemma 2.11). Thus we have

$$\begin{aligned} & \frac{g(y \cdot \eta(t), x \cdot \gamma(t)) - g(y, x \cdot \gamma(t))}{t} + \frac{f(x \cdot \gamma(t), y) - f(x, y)}{t} \\ &= g^{[1,0]}(y, \eta, t, x \cdot \gamma(t)) + f^{[1,0]}(x, \gamma, t, y) \\ &\rightarrow g^{[1,0]}(y, \eta, 0, x) + f^{[1,0]}(x, \gamma, 0, y) \end{aligned}$$

as $t \rightarrow 0$. Therefore, the derivative $df((x, y), (\gamma, \eta))$ exists and is given by

$$\begin{aligned} df((x, y), (\gamma, \eta)) &= g^{[1,0]}(y, \eta, 0, x) + f^{[1,0]}(x, \gamma, 0, y) \\ &= d^{(1,0)} g(y, x, \eta) + d^{(1,0)} f(x, y, \gamma) \\ &= d^{(0,1)} f(x, y, \eta) + d^{(1,0)} f(x, y, \gamma). \end{aligned}$$

Induction step: Now, let $2 \leq i \leq k$, $(x, y) \in U \times V$, $(\gamma_1, \eta_1), \dots, (\gamma_i, \eta_i) \in \mathfrak{L}(G \times H)$. Then for $t \neq 0$ small enough we have

$$\begin{aligned}
& \frac{1}{t} \left(d^{(i-1)} f((x \cdot \gamma_i(t), y \cdot \eta_i(t)), (\gamma_1, \eta_1), \dots, (\gamma_{i-1}, \eta_{i-1})) \right. \\
& \quad \left. - d^{(i-1)} f((x, y), (\gamma_1, \eta_1), \dots, (\gamma_{i-1}, \eta_{i-1})) \right) \\
&= \sum_{j=0}^{i-1} \sum_{I_{j,i-1}} \frac{1}{t} \left(d^{(j,i-j-1)} f(x \cdot \gamma_i(t), y \cdot \eta_i(t), \gamma_{r_1}, \dots, \gamma_{r_j}, \eta_{s_1}, \dots, \eta_{s_{i-j-1}}) \right. \\
& \quad \left. - d^{(j,i-j-1)} f(x, y, \gamma_{r_1}, \dots, \gamma_{r_j}, \eta_{s_1}, \dots, \eta_{s_{i-j-1}}) \right) \\
&= \sum_{j=0}^{i-1} \sum_{I_{j,i-1}} \frac{1}{t} \left((D_{(\gamma_{r_j}, 0)} \cdots D_{(\gamma_{r_1}, 0)} D_{(0, \eta_{s_{i-j-1}})} \cdots D_{(0, \eta_1)} f)(x \cdot \gamma_i(t), y \cdot \eta_i(t)) \right. \\
& \quad \left. - (D_{(\gamma_{r_j}, 0)} \cdots D_{(\gamma_{r_1}, 0)} D_{(0, \eta_{s_{i-j-1}})} \cdots D_{(0, \eta_1)} f)(x, y) \right).
\end{aligned}$$

Each of the maps

$$D_{(\gamma_{r_j}, 0)} \cdots D_{(\gamma_{r_1}, 0)} D_{(0, \eta_{s_{i-j-1}})} \cdots D_{(0, \eta_1)} f : U \times V \rightarrow E$$

is $C^{1,1}$ (see Remark 2.9), hence C^1 and we have

$$\begin{aligned}
& \sum_{j=0}^{i-1} \sum_{I_{j,i-1}} \frac{1}{t} \left((D_{(\gamma_{r_j}, 0)} \cdots D_{(\gamma_{r_1}, 0)} D_{(0, \eta_{s_{i-j-1}})} \cdots D_{(0, \eta_1)} f)(x \cdot \gamma_i(t), y \cdot \eta_i(t)) \right. \\
& \quad \left. - (D_{(\gamma_{r_j}, 0)} \cdots D_{(\gamma_{r_1}, 0)} D_{(0, \eta_{s_{i-j-1}})} \cdots D_{(0, \eta_1)} f)(x, y) \right) \\
& \rightarrow \sum_{j=0}^{i-1} \sum_{I_{j,i-1}} \left((D_{(\gamma_{r_j}, 0)} \cdots D_{(\gamma_{r_1}, 0)} D_{(0, \eta_i)} D_{(0, \eta_{s_{i-j-1}})} \cdots D_{(0, \eta_1)} f)(x, y) \right. \\
& \quad \left. + (D_{(\gamma_i, 0)} D_{(\gamma_{r_j}, 0)} \cdots D_{(\gamma_{r_1}, 0)} D_{(0, \eta_{s_{i-j-1}})} \cdots D_{(0, \eta_1)} f)(x, y) \right) \\
&= \sum_{j=0}^i \sum_{I_{j,i}} d^{(j,i-j)} f(x, y, \gamma_{r_1}, \dots, \gamma_{r_j}, \eta_{s_1}, \dots, \eta_{s_{i-j}})
\end{aligned}$$

as $t \rightarrow 0$ (using Proposition 2.7). Thus (3) holds, and we have

$$d^{(i)} f = \sum_{j=0}^i \sum_{I_{j,i}} d^{(j,i-j)} f \circ g_{I_{j,i}},$$

where

$$\begin{aligned}
g_{I_{j,i}} : U \times V \times \mathfrak{L}(G \times H)^i &\rightarrow U \times V \times \mathfrak{L}(G)^j \times \mathfrak{L}(H)^{i-j}, \\
(x, y, (\gamma_1, \eta_1), \dots, (\gamma_i, \eta_i)) &\mapsto (x, y, \gamma_{r_1}, \dots, \gamma_{r_j}, \eta_{s_1}, \dots, \eta_{s_{i-j}})
\end{aligned}$$

are continuous maps (see Remark 2.2). Hence f is C^k .

The linearity of the map Ψ is clear. Further, each of the maps

$$\begin{aligned} g_{I_{j,i}}^* : C(U \times V \times \mathfrak{L}(G)^j \times \mathfrak{L}(H)^{i-j}, E)_{c.o} &\rightarrow C(U \times V \times \mathfrak{L}(G \times H)^i, E)_{c.o}, \\ h &\mapsto h \circ g_{I_{j,i}} \end{aligned}$$

is continuous (see [6, Appendix A.5] or [4, Lemma B.9]), whence each of the maps

$$d^{(i)} \circ \Psi = \sum_{j=0}^i \sum_{I_{j,i}} g_{I_{j,i}}^* \circ d^{(j,i-j)}$$

is continuous. Since the topology on $C^k(U \times V, E)$ is initial with respect to the maps $d^{(i)}$, the continuity of Ψ follows. \square

Proposition 2.13. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H , let E be a locally convex space and $k, l \in \mathbb{N}_0$. If $f : U \times V \rightarrow E$ is a C^{k+l} -map, then f is $C^{k,l}$.*

Moreover, the inclusion map

$$\Psi : C^{k+l}(U \times V, E) \rightarrow C^{k,l}(U \times V, E), \quad f \mapsto f$$

is continuous and linear.

Proof. We denote by $\varepsilon_G \in \mathfrak{L}(G)$ the constant map $\varepsilon_G : \mathbb{R} \rightarrow G, t \mapsto e_G$, where e_G is the identity element of G , and $\varepsilon_H \in \mathfrak{L}(H)$ is defined analogously. Let $x \in U$, $y \in V$, $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$ and $\eta_1, \dots, \eta_j \in \mathfrak{L}(H)$ for some $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$. Then we obviously have

$$\begin{aligned} d^{(i,j)} f(x, y, \gamma_1, \dots, \gamma_i, \eta_1, \dots, \eta_j) \\ = d^{(i+j)} f((x, y), (\gamma_1, \varepsilon_H), \dots, (\gamma_i, \varepsilon_H), (\varepsilon_G, \eta_1), \dots, (\varepsilon_G, \eta_j)). \end{aligned}$$

Each of the maps

$$\begin{aligned} \rho_{i,j} : U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j &\rightarrow U \times V \times \mathfrak{L}(G \times H)^{i+j} \\ (x, y, \gamma_1, \dots, \gamma_i, \eta_1, \dots, \eta_j) &\mapsto (x, y, (\gamma_1, \varepsilon_H), \dots, (\gamma_i, \varepsilon_H), (\varepsilon_G, \eta_1), \dots, (\varepsilon_G, \eta_j)) \end{aligned}$$

is continuous (see Remark 2.2) and we have

$$d^{(i,j)} f = d^{(i+j)} f \circ \rho_{i,j}.$$

Therefore, f is $C^{k,l}$.

The linearity of the map Ψ is clear. Further, by [6, Appendix A.5] (see also [4, Lemma B.9]), each of the maps

$$\begin{aligned}\rho_{i,j}^* : C(U \times V \times \mathfrak{L}(G \times H)^{i+j}, E)_{c.o} &\rightarrow C(U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j, E)_{c.o} \\ h &\mapsto h \circ \rho_{i,j}\end{aligned}$$

is continuous, whence each of the maps

$$d^{(i,j)} \circ \Psi = \rho_{i,j}^* \circ d^{(i+j)}$$

is continuous. Hence, the continuity of Ψ follows, since the topology on the space $C^{k,l}(U \times V, E)$ is initial with respect to the maps $d^{(i,j)}$. \square

Corollary 2.14. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H , let E be a locally convex space. A map $f : U \times V \rightarrow E$ is C^∞ if and only if f is $C^{\infty, \infty}$. Moreover, the map*

$$\Psi : C^\infty(U \times V, E) \rightarrow C^{\infty, \infty}(U \times V, E), \quad f \mapsto f$$

is an isomorphism of topological vector spaces.

Proof. The assertion is an immediate consequence of Propositions 2.12 and 2.13. \square

3 The exponential law

We recall the classical Exponential Law for spaces of continuous functions, which can be found, for example, in [6, Appendix A.5]:

Proposition 3.1. *Let X_1, X_2, Y be topological spaces. If $f : X_1 \times X_2 \rightarrow Y$ is a continuous map, then also the map*

$$f^\vee : X_1 \rightarrow C(X_2, Y)_{c.o}, \quad x \mapsto f^\vee(x) := f(x, \bullet)$$

is continuous. Moreover, the map

$$\Phi : C(X_1 \times X_2, Y)_{c.o} \rightarrow C(X_1, C(X_2, Y))_{c.o}, \quad f \mapsto f^\vee$$

is a topological embedding.

If X_2 is locally compact or $X_1 \times X_2$ is a k -space, or $X_1 \times X_2$ is a $k_{\mathbb{R}}$ -space and Y is completely regular, then Φ is a homeomorphism.

The following terminology is used here:

Remark 3.2. (a) A Hausdorff topological space X is called a k -space if functions $f : X \rightarrow Y$ to a topological space Y are continuous if and only if the restrictions $f|_K : K \rightarrow Y$ are continuous for all compact subsets $K \subseteq X$. All locally compact spaces and all metrizable spaces are k -spaces.

(b) A Hausdorff topological space X is called a $k_{\mathbb{R}}$ -space if real-valued functions $f : X \rightarrow \mathbb{R}$ are continuous if and only if the restrictions $f|_K : K \rightarrow \mathbb{R}$ are

continuous for all compact subsets $K \subseteq X$. Each k -space is a $k_{\mathbb{R}}$ -space, hence also each locally compact and each metrizable space is a $k_{\mathbb{R}}$ -space.

(c) A Hausdorff topological space X is called *completely regular* if its topology is initial with respect to the set $C(X, \mathbb{R})$. Each Hausdorff locally convex space (moreover, each Hausdorff topological group) is completely regular, see [7].

Theorem 3.3. *Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H , let E be a locally convex space and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. Then the following holds:*

(a) *If a map $f : U \times V \rightarrow E$ is $C^{k,l}$, then the map*

$$f^\vee(x) := f(x, \bullet) : V \rightarrow E, \quad y \mapsto f^\vee(x)(y) := f(x, y)$$

is C^l for each $x \in U$ and the map

$$f^\vee : U \rightarrow C^l(V, E), \quad x \mapsto f^\vee(x)$$

is C^k .

(b) *The map*

$$\Phi : C^{k,l}(U \times V, E) \rightarrow C^k(U, C^l(V, E)), \quad f \mapsto f^\vee$$

is linear and a topological embedding.

Proof. (a) We will consider the following cases:

The case $k = l = 0$: This case is covered by the classical Exponential Law 3.1.

The case $k = 0, l \geq 1$: Let $x \in U$; the map $f^\vee(x) = f(x, \bullet)$ is obviously continuous, and for $y \in V$, $\eta \in \mathfrak{L}(H)$ and $t \neq 0$ small enough we have

$$\frac{1}{t}(f^\vee(x)(y \cdot \eta(t)) - f^\vee(x)(y)) = \frac{1}{t}(f(x, y \cdot \eta(t)) - f(x, y)) \rightarrow D_{(0,\eta)}f(x, y)$$

as $t \rightarrow 0$. Thus the derivative $D_\eta(f^\vee(x))(y)$ exists and equals $D_{(0,\eta)}f(x, y) = (D_{(0,\eta)}f)^\vee(x)(y)$. Proceeding similarly, for each $j \in \mathbb{N}$ with $j \leq l$ and $\eta_1, \dots, \eta_j \in \mathfrak{L}(H)$, we obtain the derivatives

$$(D_{\eta_j} \cdots D_{\eta_1}(f^\vee(x)))(y) = (D_{(0,\eta_j)} \cdots D_{(0,\eta_1)}f)^\vee(x)(y) \quad (4)$$

The obtained differentials $d^{(j)}(f^\vee(x)) = (d^{(0,j)}f)^\vee(x) : V \times \mathfrak{L}(H)^j \rightarrow E$ are continuous, therefore $f^\vee(x)$ is C^l .

Further, by the classical Exponential Law 3.1, each of the maps

$$\begin{aligned} f^\vee : U &\rightarrow C(V, E)_{c.o.}, \quad x \mapsto f^\vee(x), \\ (d^{(0,j)}f)^\vee : U &\rightarrow C(V \times \mathfrak{L}(H)^j, E)_{c.o.}, \quad x \mapsto (d^{(0,j)}f)^\vee(x) \end{aligned}$$

is continuous, and we have $d^{(j)} \circ f^\vee = (d^{(0,j)}f)^\vee$ for all $j \in \mathbb{N}_0$ with $j \leq l$. Thus, the continuity of f^\vee follows from the fact that the topology on $C^l(V, E)$ is initial with respect to the maps $d^{(j)}$.

The case $k \geq 1, l \geq 0$: By the preceding steps, the map $f^\vee(x)$ is C^l for each $x \in U$ (with derivatives given in (4)). Now we show by induction on $i \in \mathbb{N}$ with $i \leq k$ that

$$(D_{\gamma_i} \cdots D_{\gamma_1}(f^\vee))(x) = (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f)^\vee(x) \quad (5)$$

for all $x \in U$ and $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$.

Induction start: Since f is $C^{1,0}$, by Lemma 2.11 the map $f^{[1,0]} : U^{[1]} \times V \rightarrow E$ is continuous, hence so is the map $(f^{[1,0]})^\vee : U^{[1]} \rightarrow C(V, E)_{c.o}$ (see Proposition 3.1). Let $(x, \gamma, t) \in U^{[1]}$ such that $t \neq 0$ and let $y \in V$, then we have

$$\begin{aligned} \frac{1}{t}(f^\vee(x \cdot \gamma(t))(y) - f^\vee(x)(y)) &= \frac{1}{t}(f(x \cdot \gamma(t), y) - f(x, y)) \\ &= f^{[1,0]}(x, \gamma, t, y) = (f^{[1,0]})^\vee(x, \gamma, t)(y). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{t}(f^\vee(x \cdot \gamma(t)) - f^\vee(x)) &= (f^{[1,0]})^\vee(x, \gamma, t) \\ &\rightarrow (f^{[1,0]})^\vee(x, \gamma, 0) = (D_{(\gamma,0)}f)^\vee(x) \end{aligned}$$

as $t \rightarrow 0$. Thus, $D_\gamma(f^\vee)(x)$ exists and is given by $(D_{(\gamma,0)}f)^\vee(x)$.

Induction step: Now, let $2 \leq i \leq k$, $x \in U$ and $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$. For $t \neq 0$ small enough we have

$$\begin{aligned} &\frac{1}{t}((D_{\gamma_{i-1}} \cdots D_{\gamma_1}(f^\vee))(x \cdot \gamma_i(t)) - (D_{\gamma_{i-1}} \cdots D_{\gamma_1}(f^\vee))(x)) \\ &= \frac{1}{t}((D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)}f)^\vee(x \cdot \gamma_i(t)) - (D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)}f)^\vee(x)) \end{aligned}$$

by the induction hypothesis. But the map $D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)}f : U \times V \rightarrow E$ is $C^{1,0}$ (see Remark 2.6), hence by the induction start we have

$$\begin{aligned} &\frac{1}{t}((D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)}f)^\vee(x \cdot \gamma_i(t)) - (D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)}f)^\vee(x)) \\ &\rightarrow D_{\gamma_i}((D_{(\gamma_{i-1},0)} \cdots D_{(\gamma_1,0)}f)^\vee)(x) = (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f)^\vee(x), \end{aligned}$$

which shows that the derivative $(D_{\gamma_i} \cdots D_{\gamma_1}(f^\vee))(x)$ exists and is given by $(D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f)^\vee(x)$, thus (5) holds.

From Remark 2.9, we know that each of the maps

$$D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)}f : U \times V \rightarrow E$$

is $C^{0,l}$, hence $(D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} f)^\vee(x) \in C^l(V, E)$ for each $x \in U$. Now, it remains to show that each of the maps

$$\begin{aligned} d^{(i)}(f^\vee) : U \times \mathfrak{L}(G)^i &\rightarrow C^l(V, E), \\ (x, \gamma_1, \dots, \gamma_i) &\mapsto (D_{\gamma_i} \cdots D_{\gamma_1}(f^\vee))(x) = (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} f)^\vee(x) \end{aligned}$$

is continuous. To this end, let $y \in V$, $j \in \mathbb{N}_0$ with $j \leq l$ and $\eta_1, \dots, \eta_j \in \mathfrak{L}(H)$. Then we have

$$\begin{aligned} (d^{(j)} \circ d^{(i)}(f^\vee))(x, \gamma_1, \dots, \gamma_i)(y, \eta_1, \dots, \eta_j) \\ = d^{(j)}(d^{(i)}(f^\vee)(x, \gamma_1, \dots, \gamma_i))(y, \eta_1, \dots, \eta_j) \\ = [D_{\eta_j} \cdots D_{\eta_1}[(D_{\gamma_i} \cdots D_{\gamma_1}(f^\vee))(x)]](y) \end{aligned}$$

Using (5) and (4) in turn we obtain

$$\begin{aligned} [D_{\eta_j} \cdots D_{\eta_1}[(D_{\gamma_i} \cdots D_{\gamma_1}(f^\vee))(x)]](y) \\ = [D_{\eta_j} \cdots D_{\eta_1}[(D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} f)^\vee(x)]](y) \\ = (D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} f)^\vee(x)(y). \end{aligned}$$

Finally, from Proposition 2.7 we conclude

$$\begin{aligned} (D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} f)^\vee(x)(y) \\ = (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} f)^\vee(x)(y) \\ = d^{(i,j)} f(x, y, \gamma_1, \dots, \gamma_i, \eta_1, \dots, \eta_j) \\ = (d^{(i,j)} f \circ \rho_{i,j})(x, \gamma_1, \dots, \gamma_i, y, \eta_1, \dots, \eta_j) \\ = (d^{(i,j)} f \circ \rho_{i,j})^\vee(x, \gamma_1, \dots, \gamma_i)(y, \eta_1, \dots, \eta_j), \end{aligned}$$

where each $\rho_{i,j}$ is the continuous map

$$\begin{aligned} \rho_{i,j} : U \times \mathfrak{L}(G)^i \times V \times \mathfrak{L}(H)^j &\rightarrow U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j, \\ (x, \gamma, y, \eta) &\mapsto (x, y, \gamma, \eta). \end{aligned}$$

Now, from the classical Exponential Law 3.1 follows that the maps

$$(d^{(i,j)} f \circ \rho_{i,j})^\vee : U \times \mathfrak{L}(G)^i \rightarrow C(V \times \mathfrak{L}(H)^j, E)_{c.o}$$

are continuous, and we have shown that

$$d^{(j)} \circ d^{(i)}(f^\vee) = (d^{(i,j)} f \circ \rho_{i,j})^\vee, \quad (6)$$

thus the continuity of $d^{(i)}(f^\vee)$ follows from the fact that the topology on $C^l(V, E)$ is initial with respect to the maps $d^{(j)}$, whence f^\vee is C^k .

(b) The linearity and injectivity of Φ is clear. To show that Φ is a topological embedding we will prove that the given topology on $C^{k,l}(U \times V, E)$ is initial with respect to Φ . We define the functions

$$\begin{aligned} \rho_{i,j}^* : C(U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j, E)_{c.o} &\rightarrow C(U \times \mathfrak{L}(G)^i \times V \times \mathfrak{L}(H)^j, E)_{c.o}, \\ g &\mapsto g \circ \rho_{i,j}, \end{aligned}$$

and

$$\begin{aligned} \Psi_{i,j} : C(U \times \mathfrak{L}(G)^i \times V \times \mathfrak{L}(H)^j, E)_{c.o} &\rightarrow C(U \times \mathfrak{L}(G)^i, C(V \times \mathfrak{L}(H)^j, E)_{c.o})_{c.o}, \\ g &\mapsto g^\vee \end{aligned}$$

for $i, j \in \mathbb{N}_0$ such that $i \leq k, j \leq l$. Then we have

$$(d^{(i,j)} f \circ \rho_{i,j})^\vee = (\Psi_{i,j} \circ \rho_{i,j}^* \circ d^{(i,j)})(f).$$

On the other hand, we have

$$d^{(j)} \circ d^{(i)}(f^\vee) = (C(U \times \mathfrak{L}(G)^i, d^{(j)}) \circ d^{(i)} \circ \Phi)(f),$$

where $(C(U \times \mathfrak{L}(G)^i, d^{(j)}))$ are the maps

$$\begin{aligned} C(U \times \mathfrak{L}(G)^i, C^l(V, E))_{c.o} &\rightarrow C(U \times \mathfrak{L}(G)^i, C(V \times \mathfrak{L}(H)^j, E)_{c.o})_{c.o}, \\ g &\mapsto d^{(j)} \circ g. \end{aligned}$$

Thus, from (6) follows the equality

$$C(U \times \mathfrak{L}(G)^i, d^{(j)}) \circ d^{(i)} \circ \Phi = \Psi_{i,j} \circ \rho_{i,j}^* \circ d^{(i,j)}.$$

The maps $d^{(i,j)}$, $\rho_{i,j}^*$ and $\Psi_{i,j}$ are topological embeddings (see definition of the topology on $C^{k,l}(U \times V, E)$, [6, Appendix A.5], and Proposition 3.1, respectively), hence by the transitivity of initial topologies [6, Appendix A.2] the given topology on $C^{k,l}(U \times V, E)$ is initial with respect to the maps $\Psi_{i,j} \circ \rho_{i,j}^* \circ d^{(i,j)}$. But by the above equality, this topology is also initial with respect to the maps $C(U \times \mathfrak{L}(G)^i, d^{(j)}) \circ d^{(i)} \circ \Phi$. Since $d^{(i)}$ and $C(U \times \mathfrak{L}(G)^i, d^{(j)})$ are topological embeddings (see definition of the topology on $C^l(V, E)$ and [6, Appendix A.5], respectively) we conclude from [6, Appendix A.2] that the topology on the space $C^{k,l}(U \times V, E)$ is initial with respect to Φ . This completes the proof. \square

Now, we go over to the proof of Theorem (B):

Proof of Theorem (B). We need to show that if $g \in C^k(U, C^l(V, E))$, then the map

$$g^\wedge : U \times V \rightarrow E, \quad g^\wedge(x, y) := g(x)(y)$$

(which is continuous, since the locally convex space E is completely regular and we assumed that $U \times V$ is a $k_{\mathbb{R}}$ -space, see Proposition 3.1). is $C^{k,l}$. Since $\Phi(g^\wedge) = (g^\wedge)^\vee = g$, the map Φ will be surjective, hence a homeomorphism (being a topological embedding by Theorem 3.3).

To this end, we fix $x \in U$, then $g(x) \in C^l(U, E)$ and for $y \in V$, $\eta \in \mathfrak{L}(H)$ and $t \neq 0$ small enough we have

$$\frac{1}{t}(g^\wedge(x, y \cdot \eta(t)) - g^\wedge(x, y)) = \frac{1}{t}(g(x)(y \cdot \eta(t)) - g(x)(y)) \rightarrow d(g(x))(y, \eta)$$

as $t \rightarrow 0$. Consequently $d^{(0,1)}(g^\wedge)(x, y, \eta)$ exists and equals $d(g(x))(y, \eta) = (d^{(1)} \circ g)(x)(y, \eta) = (d^{(1)} \circ g)^\wedge(x, y, \eta)$. Analogously, for $j \in \mathbb{N}_0$ with $j \leq l$ and $\eta_1, \dots, \eta_j \in \mathfrak{L}(H)$ we obtain the derivatives

$$d^{(0,j)}(g^\wedge)(x, y, \eta_1, \dots, \eta_j) = (d^{(j)} \circ g)^\wedge(x, y, \eta_1, \dots, \eta_j).$$

But for fixed $(y, \eta_1, \dots, \eta_j)$ we have

$$\begin{aligned} (d^{(j)} \circ g)^\wedge(x, y, \eta_1, \dots, \eta_j) &= (d^{(j)} \circ g)(x)(y, \eta_1, \dots, \eta_j) \\ &= (\text{ev}_{(y, \eta_1, \dots, \eta_j)} \circ d^{(j)} \circ g)(x), \end{aligned}$$

where $\text{ev}_{(y, \eta_1, \dots, \eta_j)}$ is the continuous linear map

$$\text{ev}_{(y, \eta_1, \dots, \eta_j)} : C(V \times \mathfrak{L}(H)^j, E)_{c.o} \rightarrow E, \quad h \mapsto h(y, \eta_1, \dots, \eta_j).$$

Since also $d^{(j)} : C^l(V, E) \rightarrow C(V \times \mathfrak{L}(H)^j, E)_{c.o}$ is continuous and linear, the composition $\text{ev}_{(y, \eta_1, \dots, \eta_j)} \circ d^{(j)} \circ g : U \rightarrow E$ is C^k , by Lemma 2.10. Thus for $\gamma \in \mathfrak{L}(G)$ and $t \neq 0$ small enough we obtain

$$\begin{aligned} &\frac{1}{t}(d^{(0,j)}(g^\wedge)(x \cdot \gamma(t), y, \eta_1, \dots, \eta_j) - d^{(0,j)}(g^\wedge)(x, y, \eta_1, \dots, \eta_j)) \\ &= \frac{1}{t}((\text{ev}_{(y, \eta_1, \dots, \eta_j)} \circ d^{(j)} \circ g)(x \cdot \gamma(t)) - (\text{ev}_{(y, \eta_1, \dots, \eta_j)} \circ d^{(j)} \circ g)(x)) \\ &\rightarrow d(\text{ev}_{(y, \eta_1, \dots, \eta_j)} \circ d^{(j)} \circ g)(x, \gamma), \end{aligned}$$

as $t \rightarrow 0$. Thus $d^{(1,j)}(g^\wedge)(x, y, \gamma, \eta_1, \dots, \eta_j)$ is given by

$$\begin{aligned} d(\text{ev}_{(y, \eta_1, \dots, \eta_j)} \circ d^{(j)} \circ g)(x, \gamma) &= (\text{ev}_{(y, \eta_1, \dots, \eta_j)} \circ d^{(j)} \circ dg)(x, \gamma) \\ &= (d^{(j)} \circ dg)(x, \gamma)(y, \eta_1, \dots, \eta_j) \\ &= (d^{(j)} \circ dg)^\wedge(x, \gamma, y, \eta_1, \dots, \eta_j). \end{aligned}$$

Analogously, for each $i \in \mathbb{N}_0$ with $i \leq k$ and $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$ we obtain

$$d^{(i,j)}(g^\wedge)(x, y, \gamma_1, \dots, \gamma_i, \eta_1, \dots, \eta_j) = (d^{(j)} \circ d^{(i)}g)^\wedge(x, \gamma_1, \dots, \gamma_i, y, \eta_1, \dots, \eta_j).$$

To see that g^\wedge is $C^{k,l}$ we need to show that the maps

$$\begin{aligned} d^{(i,j)}(g^\wedge) : U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j &\rightarrow E, \\ (x, y, \gamma_1, \dots, \gamma_i, \eta_1, \dots, \eta_j) &\mapsto (d^{(j)} \circ d^{(i)} g)^\wedge(x, \gamma_1, \dots, \gamma_i, y, \eta_1, \dots, \eta_j) \end{aligned} \quad (7)$$

are continuous for all $i, j \in \mathbb{N}_0$ with $i \leq k, j \leq l$. To this end, consider the continuous maps

$$d^{(j)} \circ d^{(i)} g : U \times \mathfrak{L}(G)^i \rightarrow C(V \times \mathfrak{L}(H)^j, E)_{c.o.}$$

By Proposition 3.1, the maps $(d^{(j)} \circ d^{(i)} g)^\wedge : U \times \mathfrak{L}(G)^i \times V \times \mathfrak{L}(H)^j \rightarrow E$ are continuous, since E is completely regular and we assumed that $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_{\mathbb{R}}$ -space, hence the maps $d^{(i,j)}(g^\wedge)$ are continuous and g^\wedge is $C^{k,l}$. \square

Remark 3.4. Theorem (A) follows from Theorem (B), since $C^{\infty,\infty}(U \times V, E) \cong C^\infty(U \times V, E)$ as a topological vector space, by Corollary 2.14.

Corollary 3.5. *Let $U \subseteq G, V \subseteq H$ be open subsets of topological groups G and H , let E be a locally convex space and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. Assume that at least one of the following conditions is satisfied:*

- (a) $l = 0$ and V is locally compact,
- (b) $k, l < \infty$ and $U \times V \times \mathfrak{L}(G)^k \times \mathfrak{L}(H)^l$ is a $k_{\mathbb{R}}$ -space,
- (c) G and H are metrizable,
- (d) G and H are locally compact.

Then the map

$$\Phi : C^{k,l}(U \times V, E) \rightarrow C^k(U, C^l(V, E)), \quad f \mapsto f^\vee$$

is a homeomorphism.

Proof. (a) As in the proof of Theorem (B), we need to show that if $g \in C^k(U, C(V, E))$, then $g^\wedge \in C^{k,0}(U \times V, E)$. The computations of the derivatives of g^\wedge carry over (with $j = 0$), hence it remains to show that the maps $d^{(i,0)}(g^\wedge)$ in (7) are continuous for all $i \in \mathbb{N}_0$ with $i \leq k$. But since V is assumed locally compact, each of the maps $(d^{(0)} \circ d^{(i)} g)^\wedge : U \times \mathfrak{L}(G)^i \times V \rightarrow E$ is continuous by Proposition 3.1, hence so is each of the maps $d^{(i,0)}(g^\wedge)$, as required.

(b) By [9, Proposition, p.62], if $U \times V \times \mathfrak{L}(G)^k \times \mathfrak{L}(H)^l$ is a $k_{\mathbb{R}}$ -space, then so is $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ for each $i, j \in \mathbb{N}_0$ with $i \leq k, j \leq l$. Hence, Theorem (B) holds and Φ is a homeomorphism.

(c) Since G is metrizable, the space $C(\mathbb{R}, G)$ is metrizable (see [6, Appendix A.5] or [4, Lemma B.21]), whence so is $\mathfrak{L}(G) \subseteq C(\mathbb{R}, G)$ as well as $U \times \mathfrak{L}(G)^i$ for each $i \in \mathbb{N}_0, i \leq k$ as a finite product of metrizable spaces. With a similar

argumentation we conclude that also $V \times \mathfrak{L}(H)^j$ is metrizable for each $j \in \mathbb{N}_0$ with $j \leq l$, whence so is $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$. But each metrizable space is a k -space, hence a $k_{\mathbb{R}}$ -space. Therefore, Theorem (B) holds in this case and Φ is a homeomorphism.

(d) As G is locally compact, it is known that the identity component G_0 of G (being a connected locally compact subgroup of G) is a pro-Lie group (in the sense that G_0 is complete and every identity neighborhood of G_0 contains a normal subgroup N such that G/N is a Lie group, see [8, Definition 3.25]). Hence, by [8, Theorem 3.12], $\mathfrak{L}(G)$ is a pro-Lie algebra, and from [8, Proposition 3.7] follows that $\mathfrak{L}(G) \cong \mathbb{R}^I$ for some set I as a topological vector space. Since also H is assumed locally compact, for each $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ we have $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j \cong U \times V \times (\mathbb{R}^I)^i \times (\mathbb{R}^J)^j$ for some set J . Now, from [13, Theorem 5.6 (ii)] follows that $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_{\mathbb{R}}$ -space (being isomorphic to a product of completely regular locally compact spaces), whence Theorem (B) holds and Φ is a homeomorphism. \square

A Some properties of C^k - and $C^{k,l}$ -functions on topological groups

First, we prove a simple chain rule for compositions of continuous group homomorphisms and C^k -functions:

Lemma A.1. *Let G and H be topological groups, E be a locally convex space. Let $\phi : G \rightarrow H$ be a continuous group homomorphism and $f : V \rightarrow E$ be a C^k -map ($k \in \mathbb{N} \cup \{\infty\}$) on an open subset $V \subseteq H$. Then for $U := \phi^{-1}(V)$ the map*

$$f \circ \phi|_U : U \rightarrow E, \quad x \mapsto f(\phi(x))$$

is C^k .

Proof. Obviously, the map $f \circ \phi|_U$ is continuous. Now, let $x \in U$ and $\gamma \in \mathfrak{L}(G)$. For $t \neq 0$ small enough we have

$$\frac{f(\phi(x \cdot \gamma(t))) - f(\phi(x))}{t} = \frac{f(\phi(x) \cdot \phi(\gamma(t))) - f(\phi(x))}{t} \rightarrow df(\phi(x), \phi \circ \gamma)$$

as $t \rightarrow 0$, since $\phi \circ \gamma \in \mathfrak{L}(H)$, see Remark 2.2. Therefore $d(f \circ \phi|_U)(x, \gamma)$ exists and is given by $df(\phi(x), \phi \circ \gamma)$.

Repeating the above steps, we obtain for $i \in \mathbb{N}$ with $i \leq k$, $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$ the derivatives $d^{(i)}(f \circ \phi|_U)(x, \gamma_1, \dots, \gamma_i) = d^{(i)}f(\phi(x), \phi \circ \gamma_1, \dots, \phi \circ \gamma_i)$.

Now, recall that the map $\mathfrak{L}(\phi) : \mathfrak{L}(G) \rightarrow \mathfrak{L}(H)$, $\eta \mapsto \phi \circ \eta$ is continuous (Remark 2.2), whence also each of the maps

$$d^{(i)}(f \circ \phi|_U) := (d^{(i)}f) \circ (\phi|_U \times \underbrace{\mathfrak{L}(\phi) \times \dots \times \mathfrak{L}(\phi)}_{i\text{-times}}) : U \times \mathfrak{L}(G)^i \rightarrow E$$

is continuous. Hence $f \circ \phi|_U$ is C^k . \square

Lemma A.2. Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H , let $(E_\alpha)_{\alpha \in A}$ be a family of locally convex spaces with direct product $E := \prod_{\alpha \in A} E_\alpha$ and the coordinate projections $\text{pr}_\alpha : E \rightarrow E_\alpha$. For $k, l \in \mathbb{N}_0 \cup \{\infty\}$ the following holds:

(a) A map $f : U \rightarrow E$ is C^k if and only if all of its components $f_\alpha := \text{pr}_\alpha \circ f$ are C^k .

(b) A map $f : U \times V \rightarrow E$ is $C^{k,l}$ if and only if all of its components $f_\alpha := \text{pr}_\alpha \circ f$ are $C^{k,l}$.

Proof. To prove (a), first recall that because each of the projections pr_α is continuous and linear, the compositions $\text{pr}_\alpha \circ f$ are C^k if f is C^k , by Lemma 2.10 (a).

Conversely, assume that each f_α is C^k and let $x \in U$, $\gamma \in \mathfrak{L}(G)$ and $t \neq 0$ small enough. Then we have

$$\frac{1}{t}(f(x \cdot \gamma(t)) - f(x)) = \left(\frac{1}{t}(f_\alpha(x \cdot \gamma(t)) - f_\alpha(x)) \right)_{\alpha \in A}.$$

Since $\frac{1}{t}(f_\alpha(x \cdot \gamma(t)) - f_\alpha(x))$ converges to $df_\alpha(x, \gamma)$ as $t \rightarrow 0$ for each $\alpha \in A$, the derivative $df(x, \gamma)$ exists and is given by $(df_\alpha(x, \gamma))_{\alpha \in A}$.

Repeating the above steps, we obtain for $i \in \mathbb{N}$ with $i \leq k$ and $\gamma_1, \dots, \gamma_i \in \mathfrak{L}(G)$ the derivatives $d^{(i)}f(x, \gamma_1, \dots, \gamma_i) = (d^{(i)}f_\alpha(x, \gamma_1, \dots, \gamma_i))_{\alpha \in A}$, which define continuous maps

$$d^{(i)}f = \left(d^{(i)}f_\alpha \right)_{\alpha \in A} : U \times \mathfrak{L}(G)^i \rightarrow E.$$

Therefore, f is C^k .

The assertion (b) can be proven similarly, by using Lemma 2.10 (b) and showing that for all $i, j \in \mathbb{N}_0$, with $i \leq k$, $j \leq l$ we have $d^{(i,j)}f = (d^{(i,j)}f_\alpha)_{\alpha \in A}$. \square

The following lemma is a special case of Lemma 2.11:

Lemma A.3. Let $U \subseteq G$ be an open subset of a topological group G , and E be a locally convex space. A continuous map $f : U \rightarrow E$ is C^1 if and only if there exists a continuous map

$$f^{[1]} : U^{[1]} \rightarrow E$$

on the open set

$$U^{[1]} := \{(x, \gamma, t) \in U \times \mathfrak{L}(G) \times \mathbb{R} : x \cdot \gamma(t) \in U\}$$

such that

$$f^{[1]}(x, \gamma, t) = \frac{1}{t}(f(x \cdot \gamma(t)) - f(x))$$

for each $(x, \gamma, t) \in U^{[1]}$ with $t \neq 0$.

In this case we have $df(x, \gamma) = f^{[1]}(x, \gamma, 0)$ for all $x \in U$ and $\gamma \in \mathfrak{L}(G)$.

We use this lemma, as well as the analogue for C^1 -maps on locally convex spaces (which can be found in [6, Lemma 1.2.10]), for the proof of a chain rule for compositions of C^k -functions $f : G \rightarrow E$ and $g : E \rightarrow F$, which will be provided after the following version:

Lemma A.4. *Let G be a topological group, P be a topological space and E, F be locally convex spaces. Let $U \subseteq G$, $V \subseteq E$ be open subsets, and $k \in \mathbb{N} \cup \{\infty\}$. If $f : U \times P \rightarrow E$ is a $C^{k,0}$ -map such that $f(U \times P) \subseteq V$, and $g : V \rightarrow F$ is a C^k -map (in the sense of differentiability on locally convex spaces), then*

$$g \circ f : U \times P \rightarrow F$$

is a $C^{k,0}$ -map.

Proof. We may assume that k is finite and prove the assertion by induction.

Induction start: Assume that f is $C^{1,0}$, g is C^1 and let $x \in U$, $p \in P$ and $\gamma \in \mathfrak{L}(G)$. For $t \neq 0$ small enough we have

$$\begin{aligned} \frac{g(f(x \cdot (t), p)) - g(f(x, p))}{t} &= \frac{g\left(f(x, p) + t \frac{f(x \cdot \gamma(t), p) - f(x, p)}{t}\right) - g(f(x, p))}{t} \\ &= \frac{g(f(x, p) + t \cdot f^{[1,0]}(x, \gamma, t, p)) - g(f(x, p))}{t} \\ &= g^{[1]}(f(x, p), f^{[1,0]}(x, \gamma, t, p), t), \end{aligned}$$

where $g^{[1]}$, $f^{[1,0]}$ are the continuous maps from [6, Lemma 1.2.10] and Lemma 2.11. As $t \rightarrow 0$ we consequently have

$$\begin{aligned} \frac{g(f(x \cdot (t), p)) - g(f(x, p))}{t} &\rightarrow g^{[1]}(f(x, p), f^{[1,0]}(x, \gamma, 0, p), 0) \\ &= dg(f(x, p), d^{(1,0)}f(x, p, \gamma)). \end{aligned}$$

Therefore, the derivative $d^{(1,0)}(g \circ f)(x, p, \gamma)$ exists and is given by the directional derivative $dg(f(x, p), d^{(1,0)}f(x, p, \gamma))$.

Consider the continuous map

$$h : U \times P \times \mathfrak{L}(G) \rightarrow E, \quad (x, p, \gamma) \mapsto f(x, p).$$

Since $d^{(1,0)}(g \circ f)(x, p, \gamma) = (dg \circ (h, d^{(1,0)}f))(x, p, \gamma)$, the map

$$d^{(1,0)}(g \circ f) = dg \circ (h, d^{(1,0)}f) : U \times P \times \mathfrak{L}(G) \rightarrow F$$

is continuous, whence $g \circ f$ is $C^{1,0}$.

Induction step: Now, assume that f is $C^{k,0}$ and g is C^k for some $k \geq 2$. By Remark 2.6, the map $d^{(1,0)}f : U \times (P \times \mathfrak{L}(G)) \rightarrow E$ is $C^{k-1,0}$, and it is easily seen that the map $h : U \times (P \times \mathfrak{L}(G)) \rightarrow E$ defined in the induction start is $C^{k,0}$. Hence, using Lemma A.2 (b), we see that $(h, d^{(1,0)}f) : U \times (P \times \mathfrak{L}(G)) \rightarrow E \times E$

is a $C^{k-1,0}$ -map. Since $dg : V \times E \rightarrow F$ is C^{k-1} (see [6, Definition 1.3.1]), the map

$$d^{(1,0)}(g \circ f) = dg \circ (h, d^{(1,0)}) : U \times (P \times \mathfrak{L}(G)) \rightarrow F$$

is $C^{k-1,0}$, by the induction hypothesis, and from Remark 2.6 follows that $g \circ f$ is $C^{k,0}$. \square

Lemma A.5. *Let G be a topological group, E, F be locally convex spaces and $k \in \mathbb{N} \cup \{\infty\}$. Let $U \subseteq G$, $V \subseteq E$ be open subsets. If $f : U \rightarrow E$ is a C^k -map with $f(U) \subseteq V$ and also $g : V \rightarrow F$ is a C^k -map, then the map*

$$g \circ f : U \rightarrow F$$

is C^k .

Proof. We may assume that k is finite and prove the assertion by induction.

Induction start: Assume that f and g are C^1 -maps. Analogously to the preceding lemma, for $x \in U$, $\gamma \in \mathfrak{L}(G)$ and $t \neq 0$ small enough we have

$$\frac{1}{t}(g(f(x \cdot \gamma(t))) - g(f(x))) = g^{[1]}(f(x), f^{[1]}(x, \gamma, t), t),$$

with continuous maps $f^{[1]}$ as in Lemma A.3 and $g^{[1]}$ as in [6, Lemma 1.2.10]. Thus, the derivative $d(g \circ f)(x, \gamma)$ exists and we have

$$d(g \circ f)(x, \gamma) = g^{[1]}(f(x), f^{[1]}(x, \gamma, 0), 0) = dg(f(x), df(x, \gamma)).$$

Using the continuous function

$$h : U \times \mathfrak{L}(G) \rightarrow E, \quad (x, \gamma) \mapsto f(x),$$

we see that

$$d(g \circ f) = dg \circ (h, df) : U \times \mathfrak{L}(G) \rightarrow F$$

is continuous, hence $g \circ f$ is a C^1 -map.

Induction step: Now, let f and g be C^k -maps for some $k \geq 2$. Then the map $df : U \times \mathfrak{L}(G) \rightarrow E$ is $C^{k-1,0}$, by Remark 2.6, and the map $h : U \times \mathfrak{L}(G) \rightarrow E$ is obviously $C^{k,0}$. We use Lemma A.2 (b) and see that $(h, df) : U \times \mathfrak{L}(G) \rightarrow E \times E$ is a $C^{k-1,0}$ -map. By [6, Definition 1.3.1], the map $dg : V \times E \rightarrow F$ is C^{k-1} , hence by Lemma A.4, the composition

$$d(g \circ f) = dg \circ (h, df) : U \times \mathfrak{L}(G) \rightarrow F$$

is $C^{k-1,0}$, whence $g \circ f$ is C^k , by Remark 2.6. \square

Finally, the following example illustrates that the statement of Schwarz' Theorem does not hold for maps on non-abelian topological groups.

Example A.6. Consider the following subgroup G of $GL_3(\mathbb{R})$:

$$G := \left\{ x = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

(known as the Heisenberg group) and $\gamma, \eta \in \mathfrak{L}(G)$ defined as

$$\gamma(t) := \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \eta(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \quad (\forall t \in \mathbb{R}).$$

Then $G \cong \mathbb{R}^3$ via

$$\phi : G \rightarrow \mathbb{R}^3, \quad x := \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x_1, x_2, x_3).$$

Let $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a partially C^2 -map in the usual sense and define

$$f := g \circ \phi : G \rightarrow \mathbb{R}.$$

Then for each $x \in G$, the derivatives $D_\gamma f(x)$, $D_\eta f(x)$, $(D_\eta D_\gamma f)(x)$ and $(D_\gamma D_\eta f)(x)$ can be expressed using the partial derivatives of g .

First, we have

$$\begin{aligned} D_\gamma f(x) &= \lim_{t \rightarrow 0} \frac{1}{t} (f(x \cdot \gamma(t)) - f(x)) = \lim_{t \rightarrow 0} \frac{1}{t} (g(\phi(x \cdot \gamma(t))) - g(\phi(x))) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (g(x_1 + t, x_2, x_3) - g(x_1, x_2, x_3)) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (g((x_1, x_2, x_3) + t(1, 0, 0)) - g(x_1, x_2, x_3)) = \frac{\partial}{\partial x_1} g(x_1, x_2, x_3). \end{aligned}$$

Further,

$$\begin{aligned} D_\eta f(x) &= \lim_{t \rightarrow 0} \frac{1}{t} (f(x \cdot \eta(t)) - f(x)) = \lim_{t \rightarrow 0} \frac{1}{t} (g(\phi(x \cdot \eta(t))) - g(\phi(x))) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (g(x_1, x_2 + tx_1, x_3 + t) - g(x_1, x_2, x_3)) \\ &= x_1 \cdot \frac{\partial}{\partial x_2} g(x_1, x_2, x_3) + \frac{\partial}{\partial x_3} g(x_1, x_2, x_3). \end{aligned}$$

Now,

$$\begin{aligned}
(D_\eta D_\gamma f)(x) &= \lim_{t \rightarrow 0} \frac{1}{t} (D_\gamma f(x \cdot \eta(t)) - D_\gamma f(x)) \\
&= \lim_{t \rightarrow 0} \frac{1}{t} \left(\frac{\partial}{\partial x_1} g(x_1, x_2 + tx_1, x_3 + t) - \frac{\partial}{\partial x_1} g(x_1, x_2, x_3) \right) \\
&= x_1 \cdot \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1, x_2, x_3) + \frac{\partial^2}{\partial x_1 \partial x_3} g(x_1, x_2, x_3).
\end{aligned}$$

And, finally

$$\begin{aligned}
(D_\gamma D_\eta f)(x) &= \lim_{t \rightarrow 0} \frac{1}{t} (D_\eta f(x \cdot \gamma(t)) - D_\eta f(x)) \\
&= \lim_{t \rightarrow 0} \frac{1}{t} \left((x_1 + t) \cdot \frac{\partial}{\partial x_2} g(x_1 + t, x_2, x_3) + \frac{\partial}{\partial x_3} g(x_1 + t, x_2, x_3) \right. \\
&\quad \left. - x_1 \cdot \frac{\partial}{\partial x_2} g(x_1, x_2, x_3) - \frac{\partial}{\partial x_3} g(x_1, x_2, x_3) \right) \\
&= \lim_{t \rightarrow 0} \frac{x_1}{t} \left(\frac{\partial}{\partial x_2} g(x_1 + t, x_2, x_3) - \frac{\partial}{\partial x_2} g(x_1, x_2, x_3) \right) \\
&\quad + \lim_{t \rightarrow 0} \frac{1}{t} \left(\frac{\partial}{\partial x_3} g(x_1 + t, x_2, x_3) - \frac{\partial}{\partial x_3} g(x_1, x_2, x_3) \right) + \lim_{t \rightarrow 0} \frac{\partial}{\partial x_2} g(x_1 + t, x_2, x_3) \\
&= x_1 \cdot \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1, x_2, x_3) + \frac{\partial^2}{\partial x_1 \partial x_3} g(x_1, x_2, x_3) + \frac{\partial}{\partial x_2} g(x_1, x_2, x_3) \\
&= (D_\eta D_\gamma f)(x) + \frac{\partial}{\partial x_2} g(x_1, x_2, x_3).
\end{aligned}$$

Thus we see that if $\frac{\partial}{\partial x_2} g(x_1, x_2, x_3) \neq 0$, then $(D_\gamma D_\eta f)(x) \neq (D_\eta D_\gamma f)(x)$.

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